Microfoundations of heteroscedasticity in asset prices

A loss-aversion-based explanation of asymmetric GARCH models

This draft: 22/02/2016

MIHÁLY ORMOS
Department of Finance, Budapest University of Technology and Economics
Magyar tudosok krt. 2., 1117 Budapest, Hungary
Phone: +36 1 4634220; Fax: +36 1 4632745
ormos@finance.bme.hu

DUSÁN TIMOTITY
Department of Finance, Budapest University of Technology and Economics
Magyar tudosok krt. 2., 1117 Budapest, Hungary
Phone: +36 1 4634220; Fax: +36 1 4631083
timotity@finance.bme.hu

Abstract

This paper provides a theoretical explanation for the heteroscedasticity of asset returns. In line with existing empirical results, our model yields an asymmetric relationship between stock return and volatility. Based on the simple assumptions that investors behave according to Prospect Theory and are subject to mental accounting in a dynamic setting, we analytically derive the unit-root versions of two of the best fitting heteroscedasticity models (EGARCH and TGARCH). The model is supported by our empirical results from two different sides: first, analysis of individual trading data shows that investors indeed become risk-seeking right after losses and more risk-averse subsequent to gains; second, the parameter estimation of our volatility model yields the predicted negative relationship between abnormal returns and subsequent volatility.

Keywords: Asymmetric volatility; Risk seeking; Prospect theory; TGARCH; EGARCH; Volatility dynamics; Market microstructure; Heuristic-driven trader

JEL classification: C58; C93; G02; G11; G12

Acknowledgements: We gratefully acknowledge the help of Terry Odean, who provided us the individual trading dataset. We also would like to express our gratitude for the thoughtful remarks of Adam Zawadowski, which have significantly contributed to our paper. We thank Zsolt Bihary and Niklas Wagner for their comments and suggestions of at the 6th Annual Financial Market Liquidity Conference 2015 at Corvinus University of Budapest. Mihály Ormos acknowledges the support by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Dusán Timotity acknowledges the support by the Pallas Athéné Domus Scientiae Foundation.

Microfoundations of heteroscedasticity in asset prices

A loss-aversion-based explanation of asymmetric GARCH models

This draft: 22/02/2016

Abstract

This paper provides a theoretical explanation for the heteroscedasticity of asset returns. In line with existing empirical results, our model yields an asymmetric relationship between stock return and volatility. Based on the simple assumptions that investors behave according to Prospect Theory and are subject to mental accounting in a dynamic setting, we analytically derive the unit-root versions of two of the best fitting heteroscedasticity models (EGARCH and TGARCH). The model is supported by our empirical results from two different sides: first, analysis of individual trading data shows that investors indeed become risk-seeking right after losses and more risk-averse subsequent to gains; second, the parameter estimation of our volatility model yields the predicted negative relationship between abnormal returns and subsequent volatility.

Keywords: Asymmetric volatility; Risk seeking; Prospect theory; TGARCH; EGARCH; Volatility dynamics; Market microstructure; Heuristic-driven trader

JEL classification: C58; C93; G02; G11; G12

1. Introduction

Time-varying volatility (heteroscedasticity) of asset returns has attracted much research in the recent decades. Since the milestone papers of Engle (1982) and Bollerslev (1986) a great number of scholarly paper has been devoted to the topic. Their findings indicate that the phenomenon can be modeled by GARCH type models. Nevertheless, as the evidence shows below, no robust theoretical foundation has been proposed yet.

Another important aspect of the autoregression puzzle is the asymmetry in the volatility process. In particular, the phenomenon known as “asymmetric volatility” implies that changes in the price of the underlying asset are negatively correlated with the volatility of the subsequent period. Despite the wide amount of literature devoted to asymmetric volatility (Black, 1976; Christie, 1982; and Schwert, 1989), it still misses a robust explanation.

Until now three main explanations for this latter puzzle have been proposed. The first is the leverage effect noted by Black (1976), Christie (1982) and Schwert (1989). The authors assume that if the value of an equity drops, the firm becomes more leveraged, therefore, the volatility of equity returns rises according to the increased risk, hence causing the negative relationship between return and subsequent volatility. They conclude, however, that, although volatility is indeed an increasing function of financial leverage, the effect by itself is not sufficient to account for the observed negative correlation.

The second explanation labeled as the volatility feedback hypothesis states that in cases of unexpected increase in volatility (e.g. exogenous shocks), expected volatility rises accordingly, and thus increasing the required return of the given asset in line with equilibrium asset pricing models. This latter has an immediate negative impact on current stock price; therefore, it strengthens or weakens the magnitude of a previous shock subsequent to losses or gains respectively hence causing the asymmetry. Numerous papers on the topic provided evidence in support of both explanations (Pindyck, 1984; or Kim et al., 2004), yet recent studies still yield controversial results: on the one hand, Bollerslev et al. (2006) find that the analysis of high-frequency data indicates no significant volatility feedback, while on the other hand, Bekaert and Wu (2000) conclude that the leverage effect is insignificant.

The third main explanation is given by McQueen and Vorkink (2004) – their initial setting is the closest to ours – proposing that volatility autocorrelation is due to investors' inclusion of the fluctuation of prices in their perceived utility (i.e. loss-averse behavior). The authors assume that volatility increases both following gains and losses as in volatility feedback hypothesis. This assumption comes from the paper of Barberis et al. (2001), which latter provides an asset pricing interpretation of Thaler and Johnson's (1990) experiment of prospect theory in a dynamic setting. The Barberis-Huang-Santos (2001) (henceforth BHS) paper assumes that perception of losses (gains) is more (less) painful (delightful) when they are subsequent to prior losses (gains). In other words, this latter means that previous losses increase and previous gains decrease risk-aversion. However, BHS do not take into account the entire analysis of Thaler and Johnson; the authors do not focus on the finding that investors become risk-seeking following losses and risk-averse subsequent to gains if the opportunity of breaking-even is included in the choice set, which, in fact, almost always applies to asset returns. As we show in Section 2, the individual dataset we use in this paper provides support for the latter hypothesis instead of the assumption of BHS. In other words, the pattern obtained in empirical tests suggest that, in contrast to BHS and McQueen and Vorkink, volatility indeed increases following losses and decreases subsequent to gains in order to allow or prevent breaking-even respectively. In Appendix 1 we provide a detailed discussion on the empirical evidence supporting the aforementioned patterns.

It is also worth mentioning that the original setting in which autoregressive conditional heteroscedastic models were defined was the expected utility theory, that is, the dynamics of volatility (i.e. the standard deviation of asset returns in a given period) have been analyzed mostly in the setting of the mean-variance optimization of standard asset pricing models (e.g. the CAPM). However, contradictory results of this approach to utility perception have been well-documented, which would lead to biased interpretations. Therefore, we build our theoretical model on an alternative approach: the prospect theory of Kahneman and Tversky (1979).

In our paper we apply this latter approach in a dynamic interpretation along with mental accounting and derive the microfoundations of heteroscedasticity. Our main findings are that (i) previous, unexpected shocks have negative, linear effect on the investors' required return; (ii) this pattern yields a negative effect of previous market shocks on market volatility; (iii) our setting provides the microfoundations of a unit-root, asymmetric, autoregressive volatility process similar to Threshold Generalized Autoregressive Conditional Heteroscedasticity (TGARCH) and Exponential Generalized Autoregressive Conditional Heteroskedastic (EGARCH) models in discrete and continuous time respectively.

The paper is structured as follows: in section 2 the theoretical setting is derived along with the main empirical findings of behavioral patterns necessary to interpret the theory. Section 3 discusses the behavioral patterns emerging in an individual trading dataset and an empirical parameter estimation of the model through a time-series analysis of asset prices. Finally, section 4 summarizes the main conclusions of the paper and provides potential ways of further research.

2. The model

As provided in Appendix 1, the previous market return plays a dominant role in the volatility dynamics of assets and is mainly responsible for the asymmetric response to shocks. We argue that this phenomenon can be explained by applying prospect theory in a dynamic setting.

2.1. Dynamics of the required return

Assuming that investors hold portfolios similar to the market portfolio, or at least that they diversify and hence invest into multiple assets, their portfolio is highly correlated with the market. In other words, a negative or positive market shock leads to losses and gains on investors' portfolios. Thaler and Johnson (1990) show in their experimental study that in such cases (if breaking even is in the choice set) investors become risk-seeking following losses and more risk-averse subsequent to gains; they aim to avoid realizing losses (exactly as in disposition effect (Shefrin and Statman, 1985 and Odean, 1998)) and are afraid of losing previous paper gains. This behavior comes from the S-shaped value function of loss-aversion: if we include the previous outcome as a reference point, the convexity of utility perception in the domain of losses results in risk-seeking behavior as the expected utility reaches its maximum at positive risk. In this specific case, considering the previous outcome as a fixed loss would cause greater pain than aggregating in time and hoping to break even; however, realizing the previous gain yields higher expected utility than taking the risk of losing the accumulated wealth.

Therefore, mental accounting (the mental aggregation or separation of pieces of information) in a dynamic setting leads investors to aggregate in time. Hence, they aim to obtain a given reference return at each period or, at least, earn this return on average. That is, if we assume the rational expectations of outcomes as the reference point (Koszegi and Rabin, 2006), the subsequent required return is decreased by the previous abnormal return to be able to obtain the pre-defined reference return on average. Analytically the aforementioned is described with the following equation

$$\mu_t = \mu_{t-1} + \alpha(\mathrm{r}_{t-1} - \mu_{t-1}) + r_{f,t} - r_{f,t-1} \colon \alpha \in [-1,0]. \tag{1}$$

where $r_{f,t}$, r_t and μ_t stand for the risk-free rate, the portfolio return and the required portfolio return of a given investor respectively. In particular, if we assume that investors form their expectations rationally and allocate their portfolio accordingly (i.e. they choose from the efficient portfolios), μ_t represents both the required and the expected return of their portfolio as the latter two become equal. Therefore, $r_{t-1} - \mu_{t-1}$ stands for the abnormal portfolio return in the previous period, which modifies the current period required (expected) return through α. The economic interpretation of this latter variable is defined as the sensitivity of an investor to mental accounting (the aggregation of previous outcomes). It would make no sense to assume that market participants adjust the required return by more than the previous shock itself; hence, we set its lower boundary at -1. Its negative value is due to the definition: aggregating in time increases or decreases the required return subsequent to losses or gains respectively. The $r_{f,t} - r_{f,t-1}$ term is added as the correction for the change in the risk-free rate or inflation. We underline that evidence for the aforementioned patterns in the dynamics of risk-seeking, reference-dependence and framing have been bolstered by numerous studies from alternative field of science as well such as neuroeconomics (Kuhnen and Knutson, 2005) and non-human physiology (Chen et al, 2006; Lakshminarayanan et al., 2011). These findings provide support for our theoretical setting and increase the relevance and robustness of the results.

It is also worth mentioning here that autoregressive conditional heteroscedasticity (henceforth ARCH) models (Engle, 1982) were created in the setting of standard asset pricing models that are based on the expected utility theory (EUT); however, the latter would not yield the behavior described above. In contrast to prospect theory, EUT assumes concave utility in all domains of wealth, and therefore, would never induce risk-seeking behavior following losses. Therefore, the aforementioned behavior cannot be analyzed in a standard asset pricing structure; hence in order to give a coherent setting, the following section provides the definition of the risk-return relationship in prospect theory.

2.2. The mean-volatility relationship in prospect theory

The application of prospect theory in asset pricing attracted close attention in behavioral finance. Out of these, we discuss the most relevant findings that are related to our model. Levy and Levy (2004) argues that the mean-variance optimization of standard asset pricing models applies to prospect theory as well. In particular, they find that the prospect theory efficient set is a subset of the mean-variance efficient frontier and even by including probability distortion, the two sets almost coincide. Their results are confirmed and extended to asset pricing models in the paper of De Giorgi et al. (2003) and Barberis and Huang (2008). The latter papers show that if the financial market equilibrium exists then the security market line theorem of CAPM holds under cumulative prospect theory as well. This finding also means that diversifying investors hold portfolios from the capital market line, and therefore, the relationship between volatility and expected return is linear for efficient portfolios.

Adding this linearity to the theory of the inclusion of previous gains and losses (as in Eq. (1)) leads to linearly decreased and increased portfolio volatility subsequent to gains and losses respectively.

2.3. The dynamics of portfolio volatility

In the followings, we present an analytical derivation of the dynamics of volatility. We define the intertemporal change of volatility in Eq. (3) and (4). Here we assume that in an equilibrium setting the price of risk does not change over time; nonetheless, the required return is not constant but follows the dynamics of

$$\mu_t = r_{f,t} + \beta\sigma_t = \mu_{t-1} + \alpha(\mathrm{r}_{t-1} - \mu_{t-1}) + r_{f,t} - r_{f,t-1} =$$
$$= r_{f,t} + \beta\sigma_{t-1} + \alpha\left(\mathrm{r}_{t-1} - r_{f,t-1} - \beta\sigma_{t-1}\right). \tag{3}$$

Here we applied the aforementioned linearity between risk and expected return of the CAPM setting. As long as we assume that investors hold well-diversified portfolios, only systematic risk is priced; therefore, σ_t stands for the market-related portfolio risk (henceforth volatility). β represents the slope of capital market line or the price of risk. The economic interpretation of

Eq. (3) is that subsequent to losses investors aim to obtain higher expected return; however, according to equilibrium pricing, they can only achieve their goal by investing in riskier assets or increasing leverage. Solving the latter equation for the dynamics of volatility yields

$$\sigma_t = \sigma_{t-1} + \frac{\alpha}{\beta}\left(\mathrm{r}_{t-1} - r_{f,t-1} - \beta\sigma_{t-1}\right) = \sigma_{t-1} + \frac{\alpha}{\beta}e_{t-1} =$$

$$= \sigma_{t-1} + \frac{\alpha}{\beta}\sigma_{t-1}\Delta W_{t-1} = \sigma_{t-1}\left(1 + \frac{\alpha}{\beta}\Delta W_{t-1}\right), \tag{4}$$

where e_{t-1} and ΔW_{t-1} represent an error term and the change in the standard Wiener process in discrete time. Eq. (4) reveals that σ_t follows a unit-root process with constant conditional mean, that is

$$E[\sigma_{t+\tau}|\mathcal{F}_t] = \sigma_t + E\left[\sum_{i=t}^{t+\tau-1}\frac{\alpha}{\beta}\sigma_i\Delta W_i\,|\mathcal{F}_t\right] = \sigma_t + \frac{\alpha}{\beta}\sum_{i=t}^{t+\tau-1}E[\sigma_i|\mathcal{F}_t]E[\Delta W_i|\mathcal{F}_t] = \sigma_t + \frac{\alpha}{\beta}\Delta W_t \tag{5}$$

where $\mathcal{F}_t$ stands for the filtration (information available) at time *t*. Here, the separation of contemporaneous volatility and noise requires the assumption that they are uncorrelated (only the delayed response yields a negative correlation). According to Eq. (5), the volatility process seems to be valid and realistic in the sense that periodical volatility tends to remain in a finite interval over a long horizon; it converges neither to infinity nor to zero. Furthermore, Eq. (4) reveals another interesting pattern: it is very similar to the Threshold Generalized Autoregressive Conditional Heteroscedasticity (henceforth TGARCH) model introduced by Zakoian (1994) that is one of the most accurate heteroscedasticity models based on goodness-of-fit tests (Awartani and Corradi, 2005; Tavares et al., 2008). In particular, TGARCH models are defined as

$$\sigma_t = K + \delta\sigma_{t-1} + \alpha^+ {e_{t-1}}^+ + \alpha^- {e_{t-1}}^- \tag{6}$$

where ${e_{t-1}}^+ = \begin{cases} e_{t-1}\ if\ e_{t-1} > 0 \\ 0\ if\ e_{t-1} \leq 0 \end{cases}$ and ${e_{t-1}}^- = \begin{cases} e_{t-1}\ if\ e_{t-1} \leq 0 \\ 0\ if\ e_{t-1} > 0 \end{cases}$. Therefore, the special case of Eq. (4) implies that $K = 0$, $\delta = 1$ and $\alpha^+ = \alpha^- = \frac{\alpha}{\beta}$. Effects of previous gains and losses could be handled separately in Eq. (4) as well by using different α^+ and α^-; however, as we show below, previous gains play only a much less significant role in the asymmetric effect on volatility. Nevertheless, distinct α^+ and α^- would also have a reasonable economic interpretation: considering that extreme gains do not cause a negative required return, that is, investors cannot and will not invest in assets with negative expected return irrespective of the previous outcomes, gains should have a less significant effect on the subsequently required return, therefore, α^+ should differ from α^-.

Another interpretation of Eq. (4) leads to another well-fitting, asymmetric GARCH model: the Exponential Generalized Autoregressive Conditional Heteroscedasticity (EGARCH) by Nelson, (1991). Dividing by σ_{t-1} and taking the natural logarithms of both sides yields

$$\ln\sigma_t = \ln\sigma_{t-1} + \ln\left(1 + \frac{\alpha}{\beta}\Delta W_{t-1}\right). \tag{7}$$

Taking the Taylor approximation around $\Delta W_{t-1} = 0$ then gives

$$\ln\sigma_t = \ln\sigma_{t-1} + \frac{\alpha}{\beta}\Delta W_{t-1} - \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^2 {\Delta W_{t-1}}^2 + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n!}\left(\frac{\alpha}{\beta}\right)^n {\Delta W_{t-1}}^n. \tag{8}$$

Due to the well-known property of the Wiener process, as Δt approaches to zero (the continuous time version is considered) third and higher order polynomials of ΔW_t vanish and ${\Delta W_t}^2 = \Delta t$. Therefore, the continuous time version of Eq. (7) can be written as

$$\ln\sigma_t = \ln\sigma_{t-1} + \frac{\alpha}{\beta}dW_{t-1} - \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^2 dt, \tag{9}$$

or by multiplying both sides by 2

$$\ln{\sigma_t}^2 = \ln{\sigma_{t-1}}^2 + 2\frac{\alpha}{\beta}dW_{t-1} - \left(\frac{\alpha}{\beta}\right)^2 dt. \tag{10}$$

The similarity to EGARCH comes from its definition of

$$\ln{\sigma_t}^2 = \omega + \beta_1[\theta dW_{t-1} + \lambda(|dW_{t-1}| - E|dW_{t-1}|)] + \alpha_1 \ln{\sigma_{t-1}}^2, \tag{11}$$

where $\omega = -\left(\frac{\alpha}{\beta}\right)^2 dt$, $2\frac{\alpha}{\beta} = \beta_1\theta$, $\lambda = 0$ and $\alpha_1 = \alpha$ yields exactly Eq. (10). The unit-root, constant conditional mean property of Eq. (10) is again found by applying Itō's lemma for

$$x_t \equiv \ln{\sigma_t}^2, dx_t = 2\frac{\alpha}{\beta}dW_t - \left(\frac{\alpha}{\beta}\right)^2 dt. \tag{12}$$

Then the inverse function is defined as

$$\sigma_t = e^{0.5x_t}. \tag{13}$$

By Itō's lemma

$$d\sigma_t = \left[-\left(\frac{\alpha}{\beta}\right)^2 \frac{\partial\sigma_t}{\partial x_t} + \frac{1}{2}\left(2\frac{\alpha}{\beta}\right)^2 \frac{\partial^2\sigma_t}{\partial {x_t}^2}\right] + 2\frac{\alpha}{\beta}\frac{\partial\sigma_t}{\partial x_t}dW_t =$$

$$= \left[-\left(\frac{\alpha}{\beta}\right)^2 0.5\sigma_t + \frac{1}{2}\left(2\frac{\alpha}{\beta}\right)^2 0.25\sigma_t\right] + 2\frac{\alpha}{\beta}0.5\sigma_t dW_{t-1} = \frac{\alpha}{\beta}\sigma_t dW_t. \qquad (14)$$

Again, the correlation between concurrent volatility and noise has zero expected value, therefore, the conditional mean is constant regardless of the length of delay. In conclusion, the TGARCH and EGARCH models are implications of prospect theory in a dynamic setting and they represent the underlying volatility process in discrete and continuous time respectively.

2.4. The dynamics of market volatility

We have derived so far the change of investors' risk attitude and the dynamics of the volatility of their portfolios. However, reasons behind the change of market volatility have not yet been covered. In this section, we propose an explanation for the positive relationship between the dynamics of market volatility and the riskiness of investors' portfolio based on a simple market microstructural idea.

As discussed above, mental accounting leads to a clear pattern in investors choice that depends on the previous unexpected price shock: losses increase the subsequent demand for risky assets, whereas, gains reduce their demand. If we stick to the idea that, as assumed in Section 2.2, investors hold the market portfolio or at least a well-diversified one that is correlated with the market, one can clearly see the following market microstructural situation: in line with the model of Glosten and Milgrom (1985) we find informed and uninformed traders in the market with probabilities π and $(1-\pi)$ that place market orders. In their model the informed investors know the exact value of an asset that can be either high (v^H) or low (v^L) and place their orders accordingly. Other participants of the market, such as the specialists that provide the liquidity by placing limit orders (thus define the spread) know only the probability of the true value that is $P(v = v^H) = \theta$ and $P(v = v^L) = 1 - \theta$. Uninformed investors place their buy and sell orders completely randomly; hence, the probabilities of buy and sell orders coming from uninformed traders are equal ($P = 0.5$).

Therefore, the profit of specialists is generated by the losses on transactions with informed investors and gains on transactions with uninformed investors. If we assume that the market is competitive, their zero expected profit criteria for transactions at the buy limit price and at the sell limit price yield the equilibrium ask and bid prices respectively (and the spread as their difference).

Moreover, if we introduce the pattern discussed in the previous sections, the spread changes in the following way: let us assume that, based on mental accounting heuristic, there is a new type of investors in addition to informed and uninformed traders, the heuristic-driven investor. This latter definition is not new in related literature: although, according to the pioneering papers of Glosten and Milgrom (1985) and Kyle (1985) uninformed traders are defined as those who do not possess fundamental information on assets, irrespective of their motives, a definition similar to our setting has already appeared in the paper of Bloomfield et al. (2009b), in which uninformed investors can have other trading motives than fundamental (e.g. behavioral). In their study the similar three-class distinction of investors is analyzed, where informed and uninformed investors and liquidity traders are present. The liquidity trader, however, may follow a behavioral pattern according to the dynamics of liquidity demand we have discussed so far; hence, we call this class the heuristic-driven trader.

Turning back to the introduction of such traders in the equilibrium criteria, let π and δ and $(1 - \pi - \delta)$ stand for the shares of informed, heuristic-driven and uninformed traders (the probability of their trades). Then, subsequent to a negative market shock, the zero profit criteria of specialists at the ask and bid prices can be defined as

$$\theta\pi(a - v^H) + \delta(a - v) + 0.5(1 - \pi - \delta)(a - v) = 0\,, \qquad (15)$$

$$(1-\theta)\pi(\,v^L-b)+0.5(1-\pi-\delta)(v-b)=0\,, \tag{16}$$

Then the ask price is given as

$$\frac{\theta\pi v^H+0.5(1-\pi+\delta)v}{\theta\pi+0.5(1-\pi+\delta)}=v+\frac{\theta\pi\left(v^H-v\right)}{\theta\pi+0.5(1-\pi+\delta)}=v+\frac{\theta\pi(1-\theta)\left(v^H-v^L\right)}{\theta\pi+0.5(1-\pi+\delta)} \tag{17}$$

whereas the bid price follows

$$\frac{(1-\theta)\pi v^L+0.5(1-\pi-\delta)v}{(1-\theta)\pi+0.5(1-\pi-\delta)}=v+\frac{(1-\theta)\pi\left(v^L-v\right)}{(1-\theta)\pi+0.5(1-\pi-\delta)}=v-\frac{\theta\pi(1-\theta)\left(v^H-v^L\right)}{(1-\theta)\pi+0.5(1-\pi-\delta)} \tag{18}$$

One can clearly see the economic processes underlying in the aforementioned formulas: if herustic-driven traders are present the midprice differs from the expected value. Subsequent to a negative shock, the δ proportion of investors place buy orders at the ask price; however, they do not form supply at the bid price. Furthemore, their uninformed trades contribute positively to the profit; therefore, the equilibrium ask price declines as in Eq. (17). Still, their existence lowers the proportion of uninformed investors; hence, the equilibrium bid price declines as well as in Eq. (18). Although, both the ask and bid prices decline, the zero profit remains intact due to the modified probabilities of incoming buy and sell orders.

Then, the spread in competitive equilibrium can be defined as

$$S_-=\frac{\theta\pi(1-\theta)\left(v^H-v^L\right)}{\theta\pi+0.5(1-\pi+\delta)}+\frac{\theta\pi(1-\theta)\left(v^H-v^L\right)}{(1-\theta)\pi+0.5(1-\pi-\delta)}=\frac{\theta\pi(1-\theta)\left(v^H-v^L\right)}{[\theta\pi+0.5(1-\pi+\delta)][(1-\theta)\pi+0.5(1-\pi-\delta)]} \tag{19}$$

where S_- stands for the spread subsequent to a negative market shock. The spread following positive market shocks is similar except for the sign of δ:

$$S_+ = \frac{\theta\pi(1-\theta)(v^H - v^L)}{[\theta\pi+0.5(1-\pi-\delta)][(1-\theta)\pi+0.5(1-\pi+\delta)]}. \qquad (20)$$

Let the spread be defined as a function of Δ where

$$\Delta = \begin{cases} \delta \; for \; negative \; previous \; shocks \\ -\delta \; for \; positive \; previous \; shock \end{cases},$$

$$S(\Delta) = \frac{\theta\pi(1-\theta)(v^H - v^L)}{[\theta\pi+0.5(1-\pi+\Delta)][(1-\theta)\pi+0.5(1-\pi-\Delta)]}. \qquad (21)$$

Then $S_- > S_+$ if and only if $S(|\Delta|) > S(-|\Delta|)$. As the numerator takes on a constant value in the function, we focus on the denominator value $f(\Delta)$. Then, $S_- > S_+$ if and only if $f(|\Delta|) < f(-|\Delta|)$, where

$$f(\Delta) = [\theta\pi + 0.5(1-\pi+\Delta)][(1-\theta)\pi + 0.5(1-\pi-\Delta)] \qquad (22)$$

is a concave, second order polynomial function of Δ. If and only if the maximum place of this function is reached in its negative domain, then $f(|\Delta|) < f(-|\Delta|)$ is always true. Therefore, it is enough to test whether

$$\underset{\Delta}{\operatorname{argmax}} f(\Delta) < 0. \qquad (23)$$

According to the first order condition

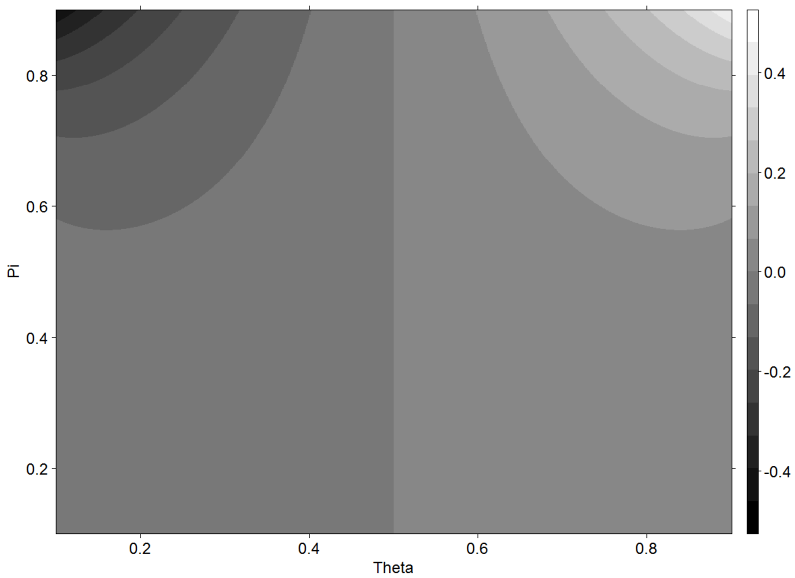
$$0.5[(1-\theta)\pi + 0.5(1-\pi-\Delta)] - 0.5[\theta\pi + 0.5(1-\pi+\Delta)] = 0\,,$$

$$\Delta = (1 - 2\theta)\pi\,. \tag{24}$$

Hence, if and only if $\theta > 0.5$, then $\underset{\Delta}{\operatorname{argmax}} f(\Delta) < 0$, $f(|\Delta|) < f(-|\Delta|)$, $S(|\Delta|) > S(-|\Delta|)$ and $S_{-} > S_{+}$. In other words, if the probability of a subsequent higher value is greater than that of a lower value, then spread is greater subsequent to a negative shock than it is following a positive shock.

The economic intuition behind an average $\theta > 0.5$ is simple as the growth of value is one of the basic assumptions in analyzing capital markets. This greater probability of a higher value is confirmed by empirical studies as well: although the authors apply a bit different methodology, Easley et al. (2002) and Brennan et al. (2014) measure the probability of an increase in the value to be $P(v|v = v^H) = 0.67$ and 0.614. The aforementioned pattern is represented in Figure 1, where the spread (defined as in Eq. (21)) is shown as a function of θ and π: the calculation is made for the constraints of $\theta, \pi \in [0.1, 0.9]$, $\delta = 0.1$, $v^H - v^L = 1$. Irrespective of probability of informed trading, π (shown in the vertical axis), the difference of spread is clearly positive for probability of higher value (shown in the horizontal axis), $\theta > 0.5$ and negative for $\theta < 0.5$ if the probability of heuristic-driven investors is positive.

Figure 1: The formation of spread if heuristic-driven trading is present



Notes: The figure represents the spread values according to the colorbar to the right. The axes, Theta and Pi stand for the probability of high value and the probability of informed traders. The difference between high and low values, $v^H - v^L$ is set to one. The probability of heuristic-driven traders is defined by $\delta = 0.1$.

In conclusion, we argue that, on average, the spread increases subsequent to losses and decreases subsequent to gains. Moreover, considering that continuous market orders at the ask and bid prices define the standard deviation of price changes, our explanation clearly implies that previous positive (negative) shocks decrease (increase) both the spread and the volatility accordingly.

Related literature provides further support to our aforementioned reasoning. Park and Sabourian (2011) analyze a similar setting based on the Glosten-Milgrom model and find that people act as contrarian if their information leads them to concentrate on middle values. Kaniel et al. (2008), Choe et al. (1999), Grinblatt and Keloharju (2000, 2001), Richards (2005), Bloomfield et al. (2009a) also confirm the existence of such contrarian traders. Moreover, according to Lof (2014), the introduction of contrarian trading in asset pricing models dramatically increases the predictive power of the models. Furthermore, our former, mental accounting-based explanation for the contrarian activity is supported by Yao and Li (2013) who argue that prospect theory investors can behave as contrarian noise traders in a market, while

Kadous et al. (2014) finds that investors act as contrarians if and only if they have held in the past the particular asset that they buy in the subsequent period; this latter provides evidence that mental accounting and prospect theory are indeed responsible for the negative feedback trading instead of an alternative exogenous factor. For the well-documented, significant, positive relationship between spread and price volatility see Hussain (2011) Wang and Yau (2000), Wyart et al. (2008).

3. Empirical results

In this section we present empirical results supporting our theory in two different ways: first, investors' dynamic behavior is tested on a large sample containing individual trading data; second, an empirical parameter estimation of our volatility model is provided using CRSP database consisting of the daily log-returns of the Standard and Poor's 500 index member listed on 10 September, 2014. The analyzed period covers 21 years from 10 September 1993 to 10 September 2014.

3.1. Patterns in intertemporal choice

In the followings, we provide empirical support to our theoretical investigation: losses and gains induce risk-seeking and more risk-averse behavior respectively. We argue that this behavior is a response to loss-aversion in a dynamic context, that is, investors are reluctant to realize losses (either physically or mentally) and try to break even in order to obtain their initial benchmark on average. According to equilibrium asset pricing, higher required return that compensates for the previous loss is only reachable by investing in assets with increased risk; therefore, combined with the change in risk attitude, losses increase the volatility of returns in the subsequent period. Gains follow the opposite pattern: investors fear of losing the previous wealth; hence, they invest into less risky portfolios since the initial benchmark level is still reachable with the latter.

The data and methodology of this analysis are as follows: Our sample is similar to that of Barber and Odean (2000) consisting of the transactions and descriptive data of 158,006 accounts at a large discount brokerage firm from January 1991 to December 1996. In this paper we aim at defining the change in the riskiness (as measured by volatility) of investors' portfolio; therefore, only common stocks investments are considered, since a meaningful amount of historical returns and realized volatility can only be calculated for these latter assets. Nevertheless, findings in this reduced sub-sample should be representative for the whole sample as the former account for 64% of the latter as measured by the number of observations. Altogether, the dataset containing at least one common stock transaction in the period includes 104,225 accounts, which can be further decomposed based on the type of the account, in which we apply cash, IRA and margin accounts as control variables, and the equity held by the related household at the end of the period. In Table 1 the descriptive statistics of these sub-samples are presented.

Table 1: Descriptive statistics of the sample

	All accounts	Cash accounts	IRA accounts	Margin accounts
Num. of accounts	104,225	22,995	37,155	10,328
Mean equity	68,293	39,859	48,988	47,953
Median equity	18,288	8,419	21,549	4,426
St. dev. of equity	300,450	129,257	129,017	247,607
Num. of trades	1,969,747	260,039	486,889	255,759
Mean number of trades	19	11	13	25

Notes: The table shows the descriptive statistics of the trading accounts included in our dataset.

In return calculations we use different types of mental frames. First, we assume that when selling occurs the profit is measured as the selling price relative to the pre-transaction average buy price of an asset. However, as the long position in an asset may include numerous buy transactions before selling the stock, we argue that if the representativity or anchoring heuristics are responsible for the change in the risk attitude, the most recent information (i.e. the price of the last buy transaction) is the main factor in utility perception. Having calculated the gain or loss, the asset into which the realized money flows in the subsequent buy transaction is defined. Related to both the bought and sold assets the variance and standard deviation of daily returns in the preceding year are calculated. Finally, based on the aforementioned parameters, regressions are estimated to analyze whether the risk of the targeted asset is driven by the previous outcome.

The first regression (first 2 columns in Table 2) applies the simple OLS estimation of the variance of the targeted asset including the profit (the return based on the average buy price) of the previous transaction as the independent variable, that is

$$\sigma_{b,i}{}^2 = \hat{\alpha} + \widehat{\beta_1}\bar{r}_{s,i} + e_i, \tag{25}$$

where $\sigma_{b,i}{}^2$ and $\bar{r}_{s,i,}$ stand for the variance of the asset in the subsequent buy transaction and the average return of the realized sell transaction of each *i* trade pair respectively.

In the second regression we test whether the change in the definition of the return increases significance and goodness-of-fit. This estimation is shown in Eq. (26) where the previous profit $r_{s,i}$ is measured as the return on the price of the last transaction.

$$\sigma_{b,i}{}^2 = \hat{\alpha} + \widehat{\beta_1} r_{s,i} + e_i, \tag{26}$$

One may argue that the variance also correlates with the risk of the sold asset as well: an investor may have a preference for risky assets, which could lead to a biased estimation of $\widehat{\beta_1}$ in the previous equation. Therefore, the third regression (Eq. (27)) includes $\sigma_{s,i}{}^2$ as the variance of the sold asset using the return on the last buy price respectively.

$$\sigma_{b,i}{}^2 = \hat{\alpha} + \widehat{\beta_1} r_{s,i} + \widehat{\beta_2} \sigma_{s,i}{}^2 + e_i, \tag{27}$$

According to equilibrium pricing, investors do require a premium for risk; thus, their expected return is different from zero. Including this finding in the fourth regression, a new definition of return may provide a better fit to utility perception: here the perceived return is defined as the deviation from the historical (one year) expected return at the last buy transaction preceding the sell transaction of an asset. In other words, we assume that investors form their non-zero expectations at the time they invest into an asset based on its performance in the past. Accordingly, as both the length of time between last buy and subsequent sell transactions and the risk of assets varies throughout the data, another adjustment is required: the expected return is not the same for each transaction; hence, we standardize the deviation from the expected return by dividing it by the number of days between the buy and sell transactions. Subsequent to this definition we use this daily average deviation from the expectation as an independent variable as in the following Eq. (28), where t_s and t_{pb} stand for the time when the sell and the previous buy transactions occurred:

$$\sigma_{b,i}{}^2 = \hat{\alpha} + \widehat{\beta_1} r_{std,s,i} + \widehat{\beta_2} \sigma_{s,i}{}^2 + e_i : r_{std,s,i} = \frac{r_{s,i} - E(r_i | t = t_{pb})}{t_s - t_{pb}} \quad (28)$$

In order to be able to distinguish effects of previous gains from losses we apply two separated variables in regression five as defined in Eq. (29):

$$\sigma_{b,i}{}^2 = \hat{\alpha} + \widehat{\beta_1} r_{-std,s,i} + \widehat{\beta_2} r_{+std,s,i} + \widehat{\beta_3} \sigma_{s,i}{}^2 + e_i : r_{-std,s,i} = \min(r_{std,s,i}, 0), r_{+std,s,i} = \max(r_{std,s,i}, 0) \quad (29)$$

Having analyzed the effects of previous outcomes on risk attitude as measured by variance, we provide further tests that include volatility instead of the former. The importance of this additional analysis is already highlighted in section 2.2, where we discussed that asset prices in prospect theory are driven by standard deviation rather than variance. Hence, in further regressions we apply volatility as the dependent variable. The sixth regression is the same as Eq. (29) except for the previously defined change in the definition of risk.

Our extensive dataset covers further parameters related to each trading account; in particular, the equity held at the end of the period and the type of the account is included as well. In further regressions we also apply these latter measures as control variables and investigate differences between the subgroups. The seventh regression is defined as in Eq. (30), where $E_i, D_{C,i}, D_{I,i}$ and $D_{M,i}$ stand for the equity, the cash type dummy, the IRA type dummy and the margin dummy of the account related to the i^{th} transaction respectively.

$$\sigma_{b,i} = \hat{\alpha} + \widehat{\beta_1} r_{std,s,i} + \widehat{\beta_2} \sigma_{s,i} + \widehat{\beta_3} E_i + \widehat{\beta_4} D_{C,i} + \widehat{\beta_5} D_{I,i} + \widehat{\beta_6} D_{M,i} + e_i, \quad (30)$$

In regression eight we modify Eq. (30) according to Eq. (29), that is, by separately estimating the coefficients of gains and losses. Then, in subsequent estimations we apply this latter frame in subgroup estimations: in the ninth equation the effects for accounts with equity value above its median (i.e. the top 50% of investors ranked by equity value) are estimated, whereas the tenth calculates coefficients for the bottom 50%. In the last three regressions effects for subgroups with a cash, IRA and margin account types are estimated.

In Table 2 we provide the empirical results of the estimations: results for groups of regressions one to three, four to six, seven to ten and eleven to thirteen are shown in Panel A, B, C and D respectively.

Table 2: Regression results

Panel A

	Subsequent σ^2		Subsequent σ^2		Subsequent σ^2	
	Coef	p-value	Coef	p-value	Coef	p-value
(Intercept)	2.32E-03	0.0000	2.32E-03	0.0000	2.22E-03	0.0000
Average return	-8.60E-05	0.0010	-	-	-	-
Return on the last trade	-	-	-9.47E-05	0.0005	-1.09E-05	0.6885
Previous variance	-	-	-	-	4.94E-02	0.0000
Adjusted R-squared	0.0000	-	0.0000	-	0.0026	-

Panel B

	Subsequent σ^2		Subsequent σ^2		Subsequent σ	
	Coef	p-value	Coef	p-value	Coef	p-value
(Intercept)	2.21E-03	0.0000	2.17E-03	0.0000	3.00E-02	0.0000
Previous variance	2.21E-03	0.0000	4.69E-02	0.0000	-	-
Expected return			-	-	-	-
Difference of last return	-1.63E-03	0.0003	-	-	-	-
Positive diff. of last return	-	-	4.72E-03	0.0000	6.65E-03	0.0071
Negative diff. of last return	-	-	-7.78E-03	0.0000	-1.48E-02	0.0000
Previous volatility	-	-	-	-	2.32E-01	0.0000
Adjusted R-squared	0.0027	-	0.0031	-	0.0386	-

Panel C								
	Subsequent σ		Subsequent σ		Subsequent σ if Equity>=Median		Subsequent σ if Equity<Median	
	Coef	p-value	Coef	p-value	Coef	p-value	Coef	p-value
(Intercept)	3.08E-02	0	3.08E-02	0.0000	3.09E-02	0.0000	3.16E-02	0.0000
Difference of last return	-4.11E-03	0.0135	-	-	-	-	-	-
Positive diff. of last return	-	-	5.53E-03	0.0251	7.49E-03	0.0429	2.43E-03	0.4627
Negative diff. of last return	-	-	-1.35E-02	0.0000	-9.93E-03	0.0098	-1.47E-02	0.0000
Previous volatility	2.30E-01	0	2.28E-01	0.0000	1.99E-01	0.0000	2.48E-01	0.0000
Equity	-2.06E-09	0	-2.06E-09	0.0000	-1.61E-09	0.0000	-5.31E-08	0.0000
Cash dummy	-5.33E-04	0.0008	-5.25E-04	0.0010	-5.60E-04	0.0213	-9.45E-04	0.0000
IRA dummy	-1.66E-03	0	-1.67E-03	0.0000	-3.16E-03	0.0000	-1.96E-04	0.2380
Margin dummy	1.50E-03	0	1.49E-03	0.0000	2.57E-03	0.0000	1.37E-04	0.4620
Adjusted R-squared	0.0419	-	0.0419	-	0.0348	-	0.0467	-

Panel D						
	Subsequent σ for cash account		Subsequent σ for IRA account		Subsequent σ for margin account	
	Coef	p-value	Coef	p-value	Coef	p-value
(Intercept)	3.00E-02	0.0000	2.84E-02	0.0000	3.06E-02	0.0000
Positive diff. of last return	5.11E-02	0.0000	7.26E-03	0.1573	-1.28E-02	0.0388
Negative diff. of last return	-9.60E-04	0.9195	-3.50E-03	0.4974	-1.77E-02	0.0009
Previous volatility	2.44E-01	0.0000	2.60E-01	0.0000	2.66E-01	0.0000
Equity	-6.65E-09	0.0000	-4.80E-09	0.0000	-9.63E-10	0.0000
Adjusted R-squared	0.0531	-	0.0565	-	0.0479	-

Notes: The table represents regression results for equations (25) to (30) and their adjustments. The dependent variables are listed in the columns, the Coef columns represent the estimated coefficients for the parameters listed in the rows, whereas the p-value columns stand for the probability of an incorrect rejection of the zero null hypothesis.

Results of the first four regressions indicate that regardless of the type of return, the aggregate effect of previous outcomes on risk attitude is significantly negative even if the previous variance is included, which supports our theory of dynamic loss-aversion. Even though, we find a minor increase in the significance by changing the reference point from the average return to the return relative to the price of the last buy transaction first, then to the return relative to the historical expected return second, the extremely low adjusted R-squared values indicate non-linear dynamics or missing variables. Regression five yields a possible reason for this latter finding: gains and losses have a distinct effect on risk attitude, although, separating the previous outcomes by their sign does add a lot to the goodness-of-fit of the latter models.

This problem is well handled by changing the risk measure to volatility: regression six shows that the adjusted R-squared value jumps, which supports our discussion on the linear relationship between standard deviation and expected return in Section 2.2.

Results of the volatility estimation of regression seven indicate four main findings: first, the aggregate effect of previous outcomes is significantly negative again; second, equity has negative effect on risk-appetite indicating that investors holding larger amounts in capital assets invest into less risky portfolios; third, market participants with cash and retirement (IRA) accounts also avoid risk shown by their negative coefficient; fourth, margin account holders have higher appetite for risk as shown by the positive relationship between subsequent volatility and the margin dummy.

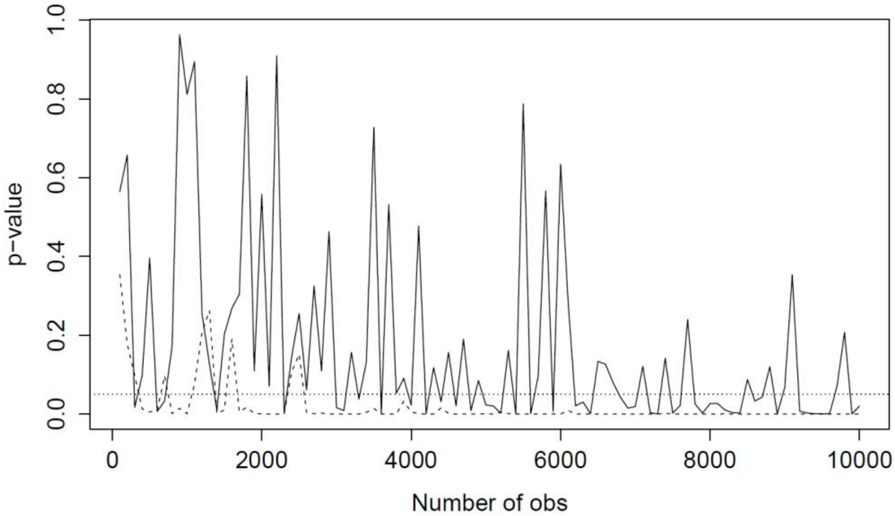
Altogether, regressions in Panel C all indicate a similar pattern as before: negative differences relative to the expected return have a significant and negative effect on the subsequent risk-appetite, whereas positive differences are either much less significant or not significant at all. In particular, regressions nine and ten show that choices of high-income investors are just as sensitive to previous outcomes as low-income investors.

In Panel D regression results show a somewhat mixed picture: although coefficients are not significant everywhere, the previous patterns apply to every subgroup except for the coefficient of the positive previous return of margin account holders. In this latter group, both previous gains and losses are significantly negative leading to lower and higher subsequent volatility respectively.

Altogether, we find similar results to the aggregated regression of Eq. (30) and its adjustment for separated gains and losses. Although, for positive deviations from the expected return we find a statistically significant positive effect on subsequent volatility, we argue that the low p-values are due to the extremely high number of observations. Following Lin et al. (2011) we provide here a test for the economic significance and analyze the effect of increasing number of observations on the significance of the coefficients of the previous outcome. According to our theoretical explanation, positive deviations from the expected return are also negatively correlated with subsequent volatility; nevertheless, since volatility is non-negative, huge realized gains lead to exactly the same portfolio choice (i.e. the risk-free asset) as a gain that is just high enough to cover two subsequent periods of the required return. Therefore, positive returns higher than a relatively small level (at least twice of the expected return) cannot be described by a linear relationship with volatility but are driven by a random process. This leads to the fact that for a reasonable number of observations, where the case of “too big to fail” does not apply, p-values of the positive coefficient should not indicate a significant effect. The last three regressions in Panel C (regressions eight to ten), in which the p-value of the coefficient of previous gains is much higher than that of losses, suggest such relationship; however, for such high number of observations a tiny effect may prove to be significant.

Hence, in Figure 2 we present the coefficient p-values as a function of the number of observations, in which regression eight is estimated with the following methodology: first, we draw 100 random sub-samples for each sample size going from 100 to 10000; second, for each of these 100 random sub-samples regression eight is estimated; third, the p-value of the 100 the coefficients obtain for each sample size is defined. We plot these p-values as a function of the sample size, where the continuous, dashed and dotted lines stand for the p-values of gains, losses and the 0.05 level respectively. One can clearly see that the coefficient of losses is always significant for sample sizes at least as high as 2500, whereas for gains p-values are above 0.05 even for sample sizes around 10,000. We argue that this finding supports our explanation of the economically insignificant effect of previous gains.

Figure 2: Coefficient p-values of previous outcomes as a function of the sample size



Notes: The figure represents the coefficient p-values of losses and gains in regression eight as the function of number of observations. We use random sampling in order to obtain lower number of observations. The solid, dashed and dotted line stand for the p-values of gains, losses and the 0.05 level respectively.

Another way to handle the non-linearity problem of previous gains would be to use a simple dummy variable for positive shocks. The intuition behind this idea is that if the expected return is relatively very small compared to the positive shocks, then, shocks exceeding this expected return have a constant effect on volatility, since investors would not and cannot reduce their required return and portfolio volatility to values below zero: they hold assets providing at least the risk-free return with zero volatility. Therefore, there is a discontinuity in the model for gains, which can be handled with the use of a dummy variable.

In the followings, we compare the results of the aforementioned model applying a dummy variable for gains and the model assuming a linear relationship between previous gains and subsequent volatility. Table 3 represents our findings.

Table 3: Regression results of volatility dynamics

	Subsequent σ			
	Coef	**p-value**	**Coef**	**p-value**
(Intercept)	3.08E-02	0.0000	3.11E-02	0.0000
Positive diff. dummy	-	-	-6.12E-04	0.0000
Positive diff. of last return	5.53E-03	0.0251	-	-
Negative diff. of last return	-1.35E-02	0.0000	-9.73E-03	0.0001
Previous volatility	2.28E-01	0.0000	2.29E-01	0.0000
Equity	-2.06E-09	0.0000	-2.07E-09	0.0000
Cash dummy	-5.25E-04	0.0010	-5.49E-04	0.0006
IRA dummy	-1.67E-03	0.0000	-1.67E-03	0.0000
Margin dummy	1.49E-03	0.0000	1.48E-03	0.0000
Adjusted R-squared	0.0419	-	0.0420	-

Notes: The table represents regression results for two regressions between previous outcomes and subsequent volatility. The dependent variable is listed in the columns, the Coef columns represent the estimated coefficients for the independent variables listed in the rows, whereas the p-value columns stand for the probability of an incorrect rejection of the zero null hypothesis.

The results indicate three important findings: first, by avoiding the discontinuity problem the regression model support our idea of a positive relationship between previous gains and volatility instead of linearity; second, this relationship becomes much more significant than in the linear model and therefore, all the variables have extremely low p-values; third, the adjusted R-squared also increases in the new model suggesting a better fit with the dummy variable. Hence, altogether the findings support the negative relationship proposed in our theoretical model.

In conclusion, we argue that the results presented in this subsection confirm the empirical validity of the behavioral side of our explanation. The aggregate coefficient of previous outcomes is negative and significant everywhere, even in regressions where other control variables are included. In particular, it seems irrelevant whether we test the effect on low- or high-income investors; the pattern emerges for all of them. Therefore, as a confirmation of the theoretical model described in Section 2, we find that previous outcomes indeed affect asset allocation and, subsequent to losses and gains, yield a money inflow into assets with higher and lower risk respectively. This finding is confirmed in existing literature on mutual fund activity as well, in which a negative relationship was found between returns and subsequent money inflows (Warther, 1995; Goetzman and Massa, 1999; Edelen and Warner, 1999) and between contemporaneous inflow of equity and bond funds (Goetzmann et al., 2000). Therefore, we argue that our model can capture and explain the unexpected changes in the demand for capital assets.

3.2. Estimating a volatility model

Based on the findings presented above, the empirical estimation of the theoretical volatility model is presented in the followings. The α and β parameters in Eq. (31), in line with Section 2.3 and 2.4, stand for the effect of intertemporal mental accounting on subsequent volatility and the price of risk respectively. Although, in the current case, α represents the aggregate effect on subsequent market volatility instead of the effect on an individual investor's portfolio risk, these coefficients are estimated according to Eq. (31), which latter is alternative form of Eq. (3). Our applied dataset covers the return and volatility time series of the daily values of the CRSP value-weighted equity index using both weekly and monthly periods covering the 21 years between 10 September 1993 and 10 September 2014. The periodic returns are defined as the sum of logarithmic daily returns. The volatility is calculated as the standard deviation of the daily returns during the given period; however, since this would show the daily volatility, it is multiplied by the ratio of the standard deviation of weekly returns divided by the standard deviation of daily returns (the adjustment to weekly from daily sampling). The estimation is based on simulating an additional error term e_t of Eq. (3), that is

$$e_t = r_t - \left(r_{f,t} + \beta\sigma_{t-1} + \alpha\left(\mathrm{r}_{t-1} - r_{f,t-1} - \beta\sigma_{t-1}\right)\right). \tag{31}$$

where $\alpha \in [-1,0]$ and $\beta \in [0,1]$. Here, the error term is not homoscedastic, therefore, we define the standardized error u_t as

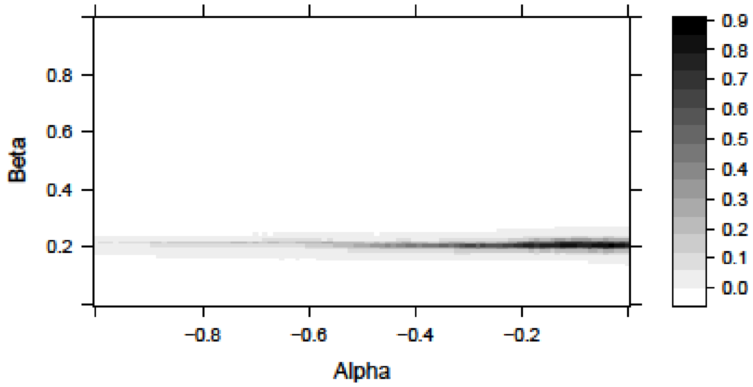
$$u_t = \frac{e_t}{\sigma_t}. \quad (32)$$

First the distribution of the error is estimated based on maximum likelihood. As Jarque-Bera tests clearly reject the null hypothesis of normality, the standardized u_t is assumed to follow a scaled Student's-t distribution with $E(u_t) = 0$. Therefore, the estimated parameters consist of the scaling *s* and degress of freedom *df.* Due to non-linear likelihood optimization, the fitted distribution is particularly sensitive to the starting values of the degrees of freedom. Hence, we provide ten estimations for each α and β pair, in which the ten starting values of the degrees of freedom are numbers equally placed in a logarithmic scale between one and the number of observations. For example, our monthly analysis includes 1061 monthly returns; therefore, the applied starting values for the degrees of freedom are 1, 2, 5, 10, 22, 48, 104, 226, 489 and 1061. The starting value of the scaling parameter is always set to one. Out of the ten estimations, the one with the highest Kolmogorov-Smirnoff p-value (the best fit) is chosen.

Second, we apply a Kolmogorov-Smirnoff test for each α and β pair to measure the significance of the difference between the empirical and estimated distribution functions. These pairs include 10201 sets consisting of the crossproducts of 101 equally placed α values between -1 and zero, and 101 equally placed β values between zero and 1. The higher the p-value, the better the fit; therefore, the $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ pair yielding the highest p-value indicates the best fit of a distribution conditional to $E(u_t) = 0$. In other words, this latter pair is considered to provide the least significant error terms.

The numerical simulation for weekly data yields $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -0.03 \\ 0.21 \end{bmatrix}$. Here, the fitted distribution has a scaling *s*=1.16 and a degree of freedom of *df=6*. The probability that we incorrectly reject the null hypothesis of the Kolmogorov-Smirnoff test is 0.8535. In Figure 3, we show the p-value (the goodness of fit) of the Kolmogorov-Smirnoff test as a function of α and β.

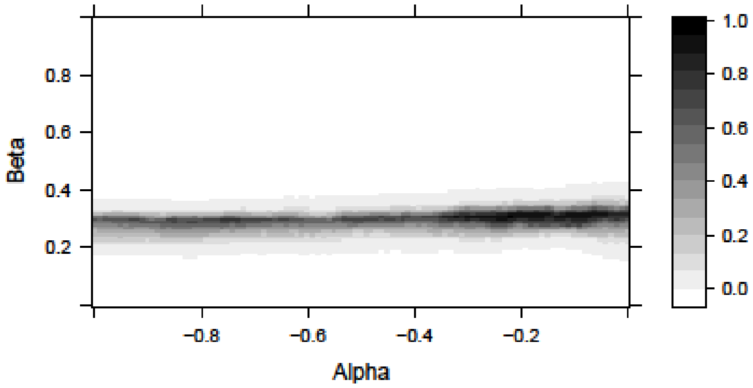
Figure 3: Kolmogorov-Smirnoff p-values in weekly analysis



Notes: The figure represents the p-values of Kolmogorov-Smirnoff tests according to the colorbar to the right. The tests apply the null hypothesis that the measured and the fitted samples come from similar distributions. We use maximum likelihood distribution fitting for the standardized error terms of Eq. (32) given the Alpha and Beta values plotted in the horizontal and vertical axes respectively. The statistics are valid for weekly sampled returns.

Our monthly analysis indicates the best fit at the $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}-0.22\\ 0.31\end{bmatrix}$ pair. The fitted distribution has a scaling *s*=1.15 and a degree of freedom of *df=225*. The Kolmogorov-Smirnoff p-value in this case test is 0.9481. Figure 4 respresents the p-value (the goodness of fit) of the Kolmogorov-Smirnoff test in the monthly results as a function of α and β.

Figure 4: Kolmogorov-Smirnoff p-values in monthly analysis



Notes: The figure represents the p-values of Kolmogorov-Smirnoff tests according to the colorbar to the right. The tests apply the null hypothesis that the measured and the fitted samples come from similar distributions. We use maximum likelihood distribution fitting for the standardized error terms of Eq. (32) given the Alpha and Beta values plotted in the horizontal and vertical axes respectively. The statistics are valid for monthly sampled returns.

Both results confirm our aforementioned reasoning for the negative effect of previous shocks on subsequent volatility; moreover, the positive relationship between risk and required return remains intact. The particularly high p-values indicate that the error terms are well fitted using scaled Student's-t distributions.

It is worth mentioning that the model presented above describes the dynamics of the volatility of the whole market. However, as presented in Appendix 1, asymmetric volatility affects individual assets as well. We argue that this phenomenon stands on the fact that market and asset returns are highly correlated, especially in periods of greater continuous shocks (e.g. the financial crisis) that affect volatility significantly. This latter correlation can be attributed to our aforementioned market microstructural reasoning: if heuristic-driven traders are present, demand for both the market portfolio and, thus, all of the risky assets increases (decreases) subsequent to negative (positive) market shocks. Moreover, the heuristic-driven demand affects positively the spread and the volatility

We present a brief correlation test between the volatilities of the index and individual assets in Table 4. The results are consistent with the proposed reasoning for the asymmetry in case of individual assets.

Table 4: Correlation between market and asset volatilities

	Weekly analysis	Monthly analysis
Positive correlations	500	499
significant at 5%	497	492
Negative correlations	0	1
significant at 5%	0	0

Notes: The table represents cross-correlation statistics between individual asset volatilities and market volatility in the same period. The columns represent the results of the weekly and monthly sampled daily volatility. The rows stand for the number of positive correlations, those which are significantly positive at 0.05 level, the number of negative correlations and those which are significantly negative at 0.05 level. The 500 pairs represent the correlations between the 500 members of the S&P500 index and the index itself.

According to weekly analysis, volatility correlation with the index is positive for all the 500 individual assets, although in three cases it is not significant. Nonetheless, these three latter assets (in particular, the equities with tickers “MNST”, “NAVI” and “NWSA”) have only become recently listed in the stock exchange, and therefore, correlation is tested on a much shorter interval than in the other cases. Hence, in these three cases the significance test yields low p-values due to the insufficient number of observations.

Applying monthly periods a similar pattern arises. Out of the 8 insignificant correlation coefficients 6 can be attributed to short available time series here as well. Altogether the positive correlation between individual assets is a robust pattern both in our weekly and monthly analysis, and hence, it is indeed a reasonable cause for the asymmetric volatility of individual assets.

4. Concluding remarks

We find that asymmetric and autoregressive volatility measured in previous empirical studies in asset pricing can be derived from and attributed to intertemporal choice of investors, assuming that they behave according to prospect theory in a dynamic setting. We show that, in contrast to most of the studies on this topic, individuals tend to become less risk-averse (or risk-seeking until a given point) and more risk-averse subsequent to losses and gains respectively, which leads to the rejection of the volatility feedback and BHS explanations for asymmetric volatility. Furthermore, we argue that the third existing explanation (the leverage effect) does not hold either, as the analysis provided yields a significant, volatility decreasing effect of both previous gains and losses of a given asset when controlling for the market return.

However, our proposed model is based on a negative relationship between market returns and market volatility, and is thus able to capture the dynamics of volatility measured empirically. Combining the linear relationship between risk and return, as presented above in detail, and the aforementioned pattern in the intertemporal choice (i.e. the required return) yields the autoregressive conditional heteroscedasticity model presented in this paper. We show that the discrete and continuous time alternatives of the main equation result in the TGARCH and EGARCH models respectively, which, in particular, are measured to be two of the regressions with the highest goodness-of-fit in most of the empirical studies. Moreover, an empirical parameter estimation in discrete time indicates that the proposed model indeed outperforms the simple random walk model, and the negative effect of previous outcomes is significant.

Potential ways of further research include various opportunities. First, an experimental analysis would be interesting to show whether these patterns are found in a laboratory environment as well if the focus is on the effect of breaking-even. Second, the influence of this behavior on asset liquidity and market microstructure could be analyzed in detail including an empirical analysis of the probability estimation of heuristic-driven traders. Third, the application of the proposed model in mathematical finance could reveal further interesting patterns; in particular, asymmetric stochastic volatility models (Heston and Nandi, 2000) in option pricing are found to provide better estimates on option prices and fit the "volatility smile" of the Black-Scholes implied volatilities, which regressions could be further improved by including the proposed model described in this paper. Finally, the introduction of cognitive research, such as the neuroeconomic approach, could reveal further underlying factors behind the behavioral patterns presented in this paper.

References

Awartani, B.M., Corradi, V. (2005). Predicting the volatility of the S&P-500 stock index via GARCH models: the role of asymmetries. International Journal of Forecasting, 21(1), 167-183.

Barber, B.M., Odean, T. (2000). Trading is hazardous to your wealth: The common stock investment performance of individual investors. Journal of Finance, 773-806.

Barberis, N., Huang, M., Santos, T. (2001). Prospect Theory and Asset Prices. Quarterly Journal of Economics, 116, 1-53

Barberis, N., Huang, M. (2008). Stocks as Lotteries: The Implications of Probability Weighting for Security Prices. American Economic Review, 98, 2066-2100.

Bekaert, G., Wu, G. (2000). Asymmetric volatility and risk in equity markets. Review of Financial Studies 13, 1–42.

Benartzi, S., Thaler, R.H. (1999). Risk aversion or myopia? Choices in repeated gambles and retirement investments. Management science, 45(3), 364-381.

Black, F. (1976). Studies of stock price volatility changes. Proceedings of the 1976 Meetings of the American Statistical Association, Business and Economical Statistics Section 177-181.

Bloomfield, R., O'hara, M., Saar, G. (2009b). How noise trading affects markets: An experimental analysis. Review of Financial Studies, 22(6), 2275-2302.

Bloomfield, R. J., Tayler, W. B., Zhou, F. H. (2009a). Momentum, reversal, and uninformed traders in laboratory markets. The Journal of Finance, 64(6), 2535-2558.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of econometrics, 31(3), 307-327.

Bollerslev, T., Litvinova, J., Tauchen, G. (2006). Leverage and volatility feedback effects in high-frequency data. Journal of Financial Econometrics, 4(3), 353-384.

Chen, M.K., Lakshminarayanan, V., Santos, L.R. (2006). How basic are behavioral biases? Evidence from capuchin monkey trading behavior. Journal of Political Economy, 114(3), 517-537.

Choe, H., Kho, B. C., Stulz, R. M. (1999). Do foreign investors destabilize stock markets? The Korean experience in 1997. Journal of Financial Economics, 54(2), 227-264.

Christie, A.A. (1982). The stochastic behavior of common stock variances – value, leverage and interest rate effects. Journal of Financial Economics 10, 407-432.

Easley, D., Hvidkjaer, S., O'hara, M. (2002). Is information risk a determinant of asset returns?. The journal of finance, 57(5), 2185-2221.

Edelen, R.M., Warner, J.B. (1999). Why are mutual fund flow and market returns related? Evidence from high-frequency data. Evidence from High-frequency Data.

Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica: Journal of the Econometric Society, 987-1007.

Goetzmann, W.N., Massa, M. (1999). Index funds and stock market growth (No. w7033). National bureau of economic research.

Goetzmann, W.N., Massa, M., Rouwenhorst, K.G. (2000). Behavioral factors in mutual fund flows. INSEAD.

Glosten, L.R., Milgrom, P.R. (1985). Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. Journal of financial economics, 14(1), 71-100.

Grinblatt, M., Keloharju, M. (2000). The investment behavior and performance of various investor types: a study of Finland's unique data set. Journal of financial economics, 55(1), 43-67.

Grinblatt, M., Keloharju, M. (2001). How distance, language, and culture influence stockholdings and trades. Journal of Finance, 1053-1073.

Hasanhodzic, J., Lo, A.W. (2011). Black's leverage effect is not due to leverage. SSRN 1762363.

Heston, S. L., Nandi, S. (2000). A closed-form GARCH option valuation model. Review of Financial Studies, 13(3), 585-625.

Hussain, S. M. (2011). The intraday behaviour of bid-ask spreads, trading volume and return volatility: evidence from DAX30. International Journal of Economics and Finance, 3(1), 23-34.

Kadous, K., Tayler, W. B., Thayer, J. M., Young, D. (2014). Individual Characteristics and the Disposition Effect: The Opposing Effects of Confidence and Self-Regard. Journal of Behavioral Finance, 15(3), 235-250.

Kaniel, R., Saar, G., Titman, S. (2008). Individual investor trading and stock returns. The Journal of Finance, 63(1), 273-310.

Kahneman, D., Tversky, A. (1979). Prospect theory: An analysis of decision under risk. Econometrica: Journal of the Econometric Society, 263-291.

Kim, C.J., Morley, J., Nelson, C. (2004). Is there a Positive Relationship between Stock Market Volatility and the Equity Premium? Journal of Money, Credit, and Banking 36, 339-360.

Kőszegi, B., Rabin, M. (2006). A model of reference-dependent preferences. The Quarterly Journal of Economics, 1133-1165.

Kuhnen, C.M., Knutson, B. (2005). The neural basis of financial risk taking. Neuron, 47(5), 763-770.

Kyle, A. S. (1985). Continuous auctions and insider trading. Econometrica: Journal of the Econometric Society, 1315-1335.

Lakshminarayanan, V.R., Chen, M.K., Santos, L.R. (2011). The evolution of decision-making under risk: framing effects in monkey risk preferences. Journal of Experimental Social Psychology, 47(3), 689-693.

Levy, H., Levy, M. (2004). Prospect theory and mean-variance analysis. Review of Financial Studies, 17(4), 1015-1041.

Lin, M., Lucas, H. C., Shmueli, G. (2011). Too Big to Fail: Larger Samples and False Discoveries. Robert H. Smith School Research Paper No. RHS, 06-068.

Lof, M. (2014). Rational Speculators, Contrarians, and Excess Volatility. Management Science.

McQueen, G., Vorkink, K. (2004). Whence GARCH? A preference-based explanation for conditional volatility. Review of Financial Studies, 17(4), 915-949.

Odean, T. (1998). Are investors reluctant to realize their losses? Journal of Finance, 53(5), 1775-1798.

Park, A., Sabourian, H. (2011). Herding and contrarian behavior in financial markets. Econometrica, 79(4), 973-1026.

Pindyck, R.S. (1984) Risk, inflation, and the stock market. American Economic Review 74, 334–351.

Richards, A. (2005). Big fish in small ponds: The trading behavior and price impact of foreign investors in Asian emerging equity markets. Journal of Financial and quantitative Analysis, 40(01), 1-27.

Shefrin, H., Statman, M. (1985). The disposition to sell winners too early and ride losers too long: Theory and evidence. The Journal of finance, 40(3), 777-790.

Schwert, G.W. (1989). Why does stock market volatility change over time? Journal of Finance 44, 1115–1153.

Tavares, A.B., Curto, J.D., Tavares, G.N. (2008). Modelling heavy tails and asymmetry using ARCH-type models with stable Paretian distributions. Nonlinear Dynamics, 51(1-2), 231-243.

Thaler, R.H., Johnson, E.J. (1990). Gambling with the house money and trying to break even: The effects of prior outcomes on risky choice. Management science, 36(6), 643-660.

Tversky, A., Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. Journal of Risk and uncertainty, 5(4), 297-323.

Wang, G. H., Yau, J. (2000). Trading volume, bid-ask spread, and price volatility in futures markets. Journal of Futures markets, 20(10), 943-970.

Warther, V.A. (1995). Aggregate mutual fund flows and security returns. Journal of financial economics, 39(2), 209-235.

Wyart, M., Bouchaud, J. P., Kockelkoren, J., Potters, M., Vettorazzo, M. (2008). Relation between bid–ask spread, impact and volatility in order-driven markets. Quantitative Finance, 8(1), 41-57.

Zakoian, J.M. (1994). Threshold heteroskedastic models. Journal of Economic Dynamics and control, 18(5), 931-955.

Yao, J., Li, D. (2013). Prospect theory and trading patterns. Journal of Banking & Finance, 37(8), 2793-2805.

Appendix 1

In the followings, we provide empirical evidence that further develops the existing explanations of asymmetric volatility effect. In theory, the volatility feedback would imply increased volatility following price shocks regardless of their sign while leverage effect would imply negative correlation with the asset return. The data used to test the hypotheses consist of the daily log-returns of members of the Standard and Poor's 500 index on 10 September, 2014. The analyzed period covers 21 years from 10 September 1993 to 10 September 2014.

Methodology

The methodology of the regression analysis is defined as follows. First, the data is pooled together, therefore, the 501 (500 members and the index itself) time-series constitute the panel. Then, two different panels are created by calculating the non-overlapping weekly and monthly returns by aggregating the daily log-returns. In the same time, a volatility panel is generated as well by measuring the standard deviation of daily returns in each weekly and monthly period for each asset. As the aim is to measure the effect of previous shocks on the change of volatility the first regression is estimated as

$$\Delta vol_{t,i} = \hat{\alpha} + \hat{\beta} r_{t-1,i} + e_{t,i}, \tag{A1}$$

where $\Delta vol_{t,i} = vol_{t,i} - vol_{t-1,i}$, $r_{t-1,i}$ and $e_{t,i}$ stand for the difference of volatility (i.e. standard deviation of daily returns in period *t*), the previous period return and the error term of the *i*-th asset respectively. One might argue that positive and negative shocks have a different effect on the change of volatility, thus causing biased results. Therefore we expand Eq. (A1) with a variable $r^+{}_{t-1}$ that takes the value of the return if it is non-negative and zero otherwise. The estimation yields the form

$$\Delta vol_{t,i} = \hat{\alpha} + \widehat{\beta_1} r_{t-1,i} + \widehat{\beta_2} r^+{}_{t-1,i} + e_{t,i}. \quad \text{(A2)}$$

In order to test the discontinuity around zero return (in the ±0.05 quantile environment of the empirical distribution), a dummy for positive values, $\hat{\alpha}^+$ is tested as well; hence

$$\Delta vol_{t,i} = \hat{\alpha} + \hat{\alpha}^+ + \widehat{\beta_1} r_{t-1,i} + \widehat{\beta_2} r^+{}_{t-1,i} + e_{t,i}. \quad \text{(A3)}$$

The existence of volatility feedback effect could be analyzed at this point by testing whether the coefficients of both previous gains and losses are significantly higher than zero. However, the interpretation of the leverage effect still might be misleading due to omitted factors. The latter explanation states that volatility is negatively correlated with previous asset returns; nonetheless, Eq. (A3) may include the contribution of other factors such as the market return as well. Hence, the following estimation reveals the clean, decomposed effect of the previous market and asset returns on volatility:

$$\Delta vol_{t,i} = \hat{\alpha} + \widehat{\beta_1} r_{t-1,i} + \widehat{\beta_2} r^+{}_{t-1,i} + \widehat{\beta_3} r_{M,t-1} + \widehat{\beta_4} r^+{}_{M,t-1} + e_{t,i}, \quad \text{(A4)}$$

where $r_{M,t-1}$ and $r^+{}_{M,t-1}$ stand for the market return and the non-negative market return respectively. One may argue that due to data pooling, the average change of volatility ($\hat{\alpha}$) of separate assets differs; thus, causing biased estimation. The representation of Eq. (A4) hence becomes

$$\Delta vol_{t,i} = \hat{\alpha}_i + \widehat{\beta_1} r_{t-1,i} + \widehat{\beta_2} {r^+}_{t-1,i} + \widehat{\beta_3} r_{M,t-1} + \widehat{\beta_4} {r^+}_{M,t-1} + e_{t,i}. \quad \text{(A5)}$$

Using the fixed effect approach that filters out individual trends, which demeans the variables with respect to time in this particular setting, yields

$$\Delta vol_{t,i} - \mathrm{E}[\Delta vol_{t,i}] = \widehat{\beta_1}(r_{t-1} - \mathrm{E}[r_{t-1}]) + \widehat{\beta_2}({r^+}_{t-1} - \mathrm{E}[{r^+}_{t-1}]) +$$
$$+\widehat{\beta_3}(r_{M,t-1} - \mathrm{E}[r_{M,t-1}]) + \widehat{\beta_4}({r^+}_{M,t-1} - \mathrm{E}[{r^+}_{M,t-1}]) + e_{t,i}. \quad \text{(A6)}$$

Empirical results

Table A1 summarizes the OLS estimation results of models from Eq. (A1) to (A6) (as indicated by numbers in the first row) using weekly statistics.

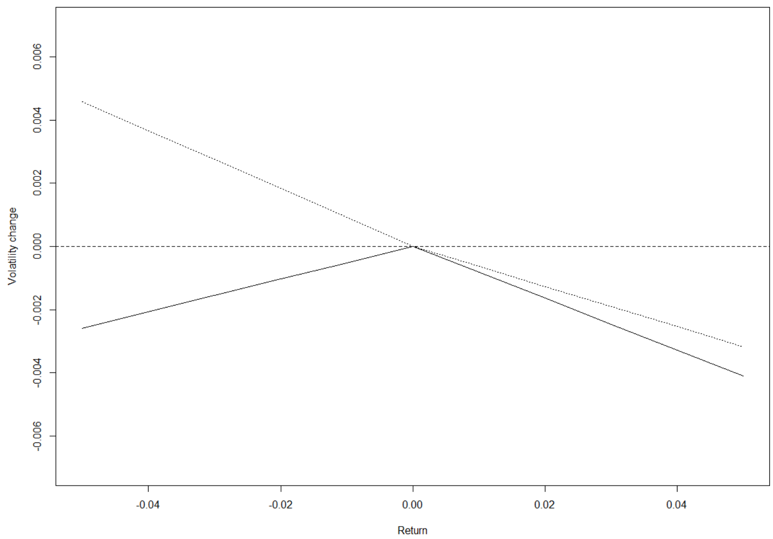
Table A1: Weekly regression results

	1		2		3		4		6	
	Coef	p-value	Coef	p-value	Coef	p-value	Coef	p-value	Coef	p-value
α	0.0000	0.0991	0.0020	0.0000	0.0014	0.0000	0.0010	0.0000	-	-
α+	-	-	-	-	-0.0002	0.3489	-	-	-	-
β1	-0.0214	0.0000	0.0306	0.0000	0.1262	0.0337	0.0544	0.0000	0.0518	0.0000
β2	-	-	-0.1070	0.0000	-0.2301	0.0045	-0.1326	0.0000	-0.1339	0.0000
β3	-	-	-	-	-	-	-0.1252	0.0000	-0.0916	0.0000
β4	-	-	-	-	-	-	0.1621	0.0000	0.0281	0.0000

Including market return as well reveals an interesting pattern: changes of volatility are negatively correlated with previous market returns, whereas we find no correlation between previous asset return and volatility (thus, volatility decreases subsequent to both gains and losses). Contrary to the volatility feedback hypothesis and McQueen and Vorkink's explanation, according to the negative aggregate effect (β1+β2) present in Eq. (A1) to (A6) for previous gains, volatility indeed decreases. Furthermore, the positive coefficients in Eq. (A2) to (A6) of β1 reject the existence of the leverage effect that would require a negative relationship between losses and the change of volatility. This latter finding is in line with the findings of Hasanhodzic and Lo (2011): they underline that leverage effect is existent in the case of all-equity-financed companies as well, and therefore, reject the hypothesis.

In order to better represent the decomposed effect, Figure A1 shows Eq. (A6) on the [-0.05,0.05] return interval where the solid and dotted lines represent the effect of previous asset and market return respectively. The dashed line stands for $\widehat{\Delta vol_{t,i}} - \mathrm{E}\left[\Delta vol_{t,i}\right] = 0$.

Figure A1: Effect of previous asset and market shocks



We argue that the effects are driven by two distinct factors. On the one hand, the previous market return has indeed a negative correlation with the volatility of an asset. As we show below, this is due to the aggregate perception of utility caused by the loss-averse behavior in a dynamic setting. In other words, investors turn to riskier assets providing higher expected return in order to be able to break even and compensate for previous losses. On the other hand, the volatility-decreasing effect of previous shocks of a given asset is simply caused by the base effect: in case of massive shocks in the asset price the contemporaneous volatility rises and, since volatility follows a strong mean-reverting process, the unusually high or low level of volatility is followed by a decrease or increase respectively. This latter proposition is supported by the results that the lag-1 autocorrelation of the volatility difference of each asset is negative and is significant at 5% in our sample: a rise in volatility is likely to be followed by a decrease.

In order to increase the robustness of these results a monthly test is also provided in Table A2, where we focus on regressions Eq. (A1) and (A6) omitting the transitory steps. The results are in line with the weekly analysis, and therefore, the pattern is robust regardless of the length of the period applied.

Table A2: Monthly regression results

	1		6	
	Coef	**p-value**	**Coef**	**p-value**
A	0.0000	0.2898	-	-
β1	-0.0080	0.0000	0.0169	0.0000
β2	-	-	-0.0423	0.0000
β3	-	-	-0.0317	0.0000
β4	-	-	0.0210	0.0000

One may argue that these results are subject to selection bias (i.e. some members existent at the end of the examination period have no trading data at the beginning): fluctuations towards the end of the period are over weighted. Therefore, another robustness test is presented in the first column of Table A3 including the 340 equities traded in both 1993 and 2014. Another argument against the latter regressions could be related to the 21 year period in use, therefore, in column 2 and 3 subperiods of 2007-2008 covering the main events of the recent financial crisis and that of 2009-2014 are estimated. These tests are provided only for weekly data in order to keep the number of observations sufficiently high and only Eq. (A6) is estimated to compare the results.

Table A3: Robustness test results

	340 members		2007-2008		2009-2014	
	Coef	**p-value**	**Coef**	**p-value**	**Coef**	**p-value**
α	-	-	-	-	-	-
β1	0.0352	0	-0.0163	0	0.0019	0.105
β2	-0.1019	0	0.01	0.008	-0.0011	0.351
β3	-0.0776	0	-0.1058	0	-0.0494	0
β4	0.016	0	-0.0552	0.007	0.0021	0.558

These results confirm that market return is negatively correlated with the subsequent change in volatility, whereas the effect of the asset return either remain the same or becomes minor in magnitude relative to that of the market return.